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Stationary flows of the parabolic potential barrier in two dimensions

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Abstract. In the two-dimensional isotropic parabolic potential barrier $V(x, y) = V_0 - m\gamma^2(x^2 + y^2)/2$, though it is a model of an unstable system in quantum mechanics, we can obtain the stationary states corresponding to the real energy eigenvalue V_0. Further, they are infinitely degenerate. For the first few eigenstates, we will find the stationary flows round a right angle that are expressed by the complex velocity potentials $W = \pm\gamma z^2/2$. From the hydrodynamical point of view vortex structures for the general solutions are also studied.

1. Introduction

It is well known that the two-dimensional harmonic oscillator (2D HO) is equivalent to the dynamical system consisting of two independent one-dimensional harmonic oscillators (1D HOs)—the energy eigenvalues of the 2D HO are given by the sum of the energy eigenvalues of the 1D HOs and the eigenstates of the system are given by the product of the eigenstates of the 1D HOs. When degenerate eigenstates of the 2D HO are superposed with suitable weights, the new states will be the eigenstates of orbital angular momentum. These results were studied a long time ago by Dirac [1].

In this paper we will investigate the two-dimensional parabolic potential barrier (2D PPB) defined by the potential energy $V(x, y) = V_0 - m\gamma^2(x^2+y^2)/2$, which is a model of an unstable system in quantum mechanics, on the same lines as the 2D HO. This model is equivalent to the dynamical system consisting of two independent one-dimensional parabolic potential barriers (1D PPBs). Note that the 1D PPB has been studied by various authors [2–12]. Discussions similar to the present model were performed by the method of complex scaling [13, 14] and also in terms of the complex harmonic oscillator [15, 16]. In the 1D PPB defined by the potential energy $V(x) = V_0/2 - m\gamma^2x^2/2$, we have shown that the energy eigenvalues are complex numbers and the corresponding eigenfunctions are expressible in terms of the generalized functions of a Gel'fand triplet [11, 12]. It should be noticed that all energy eigenvalues appear in the pairs of conjugate complexes $V_0/2 \mp \mathrm{i}(n + 1/2)\hbar\gamma$ (n a non-negative integer), which, respectively, correspond to the resonance states of decay and growth.

In two dimensions, however, the exact solutions of the eigenvalue problem have much more variety. In section 3 we will see that they are separated into four types. Two of the four types (in section 3.1), that are expressed by the generalized eigenfunctions belonging to complex (not real) energy eigenvalues, represent diverging and converging flows. In these

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two types the solutions will also be the eigenstates of orbital angular momentum. In the other two of the four types (in section 3.2) all the solutions are infinitely degenerate and involve the special solutions with the real energy eigenvalue V_0, which are stationary and do not grow or decay with time. It is a striking result that the eigenstates of the 2D PPB can involve *stationary states*, while no such state exists in the 1D PPB. This situation may be well understood in comparison with the results of classical mechanics for the 2D PPB. That is to say, for the 1D PPB, the Newton equation of motion is $\ddot{x}(t) = \gamma^2 x(t)$ and then the fundamental solutions give $x^{\pm}(t) = x^{\pm}\mathrm{e}^{\pm\gamma t}$, where $x^{\pm}$ are real numbers. These solutions, $x^{+}(t)$ and $x^{-}(t)$, respectively, represent the diverging solution (tends to infinity) and the converging one (tends to zero). In two dimensions we find two different orbits expressed by $x^{\pm}(t)/y^{\pm}(t) = \text{constant}$ and $x^{\pm}(t)y^{\mp}(t) = \text{constant}$. Now it is transparent that the former two of the four types of quantum solution of the 2D PPB correspond to the linear orbits $x^{\pm}(t)/y^{\pm}(t) = \text{constant}$ and the latter two involving the stationary states to the hyperbolic orbits $x^{\pm}(t)y^{\mp}(t) = \text{constant}$. We may say that linear orbits such that $x^{\pm}(t)$ and $y^{\pm}(t)$ go toward the origin as $t \to \mp\infty$ correspond to the states of resonance in quantum mechanics, whereas the hyperbolic orbits which cannot pass through the origin correspond to the scattering states in collision problems. We may expect that, even though the time-independent eigenfunctions expressed by the generalized functions of a Gel'fand triplet are not normalizable, such as the free particle or the stationary states of collision problems, we can obtain physical results in terms of the ratios of the probability densities or the probability currents in the initial and final states. The probability density $\rho(t, \boldsymbol{r})$ and the probability current $\boldsymbol{j}(t, \boldsymbol{r})$ of a state $\psi(t, \boldsymbol{r})$ in non-relativistic quantum mechanics are defined by [17–19]

$$\rho(t, \boldsymbol{r}) \equiv |\psi(t, \boldsymbol{r})|^2 \tag{1.1}$$

$$\boldsymbol{j}(t, \boldsymbol{r}) \equiv \mathrm{Re}\,[\psi(t, \boldsymbol{r})^*(-\mathrm{i}\hbar\nabla)\psi(t, \boldsymbol{r})]/m \tag{1.2}$$

where m is the mass of the particle. They satisfy the equation of continuity

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\boldsymbol{j} = 0. \tag{1.3}$$

However, these ρ and $\boldsymbol{j}$ are independent of the time t when the considered state is a closed state of the stable system (e.g. the HO, the hydrogen atom) or a stationary state (e.g. the free particle). For a non-stationary state of the PPB, on the other hand, the probability density (1.1) and the probability current (1.2) are not simple, i.e. they generally depend on the time t and the coordinate $\boldsymbol{r}$. We therefore need some further study to describe the stationary and non-stationary flows of the 2D PPB. We will see that some kind of *velocity* introduced in a hydrodynamical formulation of quantum mechanics can be an interesting physical quantity.

Another difference will be seen in the infinite degeneracy of the stationary states in quantum mechanics. This property enable us to write the states in many various expressions in terms of superposition of the infinitely degenerate states. In section 4.1 we will actually see that *quantized vortices*, which connect the nodal singularities of the wavefunction, appear. Furthermore, we shall also show that *complex velocity potentials* are useful for a few stationary flows in section 4.2. These considerations indicate that the hydrodynamical approach will give us an interesting insight for understanding of our results.

The connection between hydrodynamics and quantum mechanics was vigorously investigated in the earlier stage of the development of quantum mechanics [20–27]†. From the chemical side the connection between the nodes of the wavefunction and quantized vortices was carefully examined by Hirschfelder [28–31]. A review article including these works is given by [32]. It is also noticed that such a velocity is still useful in present-day quantum

† This was pointed out to us by R Jackiw.

physics [33–35]. A brief review of the *hydrodynamical formulation of quantum mechanics* is presented in section 2. The vortex structures of the 2D PPB flows solved in section 3.1 will be studied in section 4.1. In section 4.2 we shall find that, for the first few solutions of section 3.2, the quantum velocities are solenoidal, so the corresponding complex velocity potentials must exist. These complex velocity potentials for the 2D PPB describe the flows round a right angle.

2. Velocities, vortices and complex velocity potentials in quantum mechanics

We start out with the equations of hydrodynamics, consisting of Euler's equation of continuity for the density and velocity of a fluid and so on. Let us try to introduce a velocity which will be the analogue of the hydrodynamical one. In hydrodynamics [36–38], the fluid at one time can be represented by the density ρ and the fluid velocity $\boldsymbol{v}$. They satisfy Euler's equation of continuity

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{v}) = 0. \tag{2.1}$$

Comparing this equation with (1.3), we are thus led to the following definition for the quantum velocity of a state $\psi(t,\boldsymbol{r})$:

$$\boldsymbol{v} \equiv \frac{\boldsymbol{j}(t,\boldsymbol{r})}{|\psi(t,\boldsymbol{r})|^2} \tag{2.2}$$

in which $\boldsymbol{j}(t,\boldsymbol{r})$ is given by (1.2). It is stressed that this velocity is different from the eigenvalue of the velocity operator $\hat{\boldsymbol{v}} = -\mathrm{i}\hbar\nabla/m$, except for the cases where $\psi(t,\boldsymbol{r})$ is an eigenfunction of the momentum operator $\hat{\boldsymbol{p}} = -\mathrm{i}\hbar\nabla$. Note that, if we can separate variables of $\psi(t,\boldsymbol{r})$, then $\boldsymbol{v}$ does not contain the time t explicitly.

Equation (2.2) is justified from the following point of view. In semiclassical cases the time-dependent wavefunction can be written [17–19]

$$\psi(t,\boldsymbol{r}) = \sqrt{\rho}\,\mathrm{e}^{\mathrm{i}S/\hbar} \tag{2.3}$$

where ρ is the probability density (1.1) and S is the quantum analogue of the classical action, which is a real function of t and $\boldsymbol{r}$. The velocity (2.2) gives

$$\boldsymbol{v} = \nabla S/m. \tag{2.4}$$

The right-hand side is known just as the velocity in classical dynamics. Madelung [20] introduced the velocity by this relation (2.4).

We shall now consider the vorticity in quantum mechanics. It is defined, as in hydrodynamics, by

$$\boldsymbol{\omega} \equiv \nabla\times\boldsymbol{v}. \tag{2.5}$$

With the above definition we have

$$\boldsymbol{\omega} = 0 \tag{2.6}$$

for the domain in which the wavefunction does not have nodes. Using standard formulae of vector analysis, equation (2.6) is verified with the help of (2.4), in semiclassical cases. However, equation (2.6) does not hold when the velocity has singularities. From the definition (2.2), in the nodal region where the wavefunction vanishes, a vortex may exist [29, 31]. The strength of the vortex is characterized by the circulation round a closed contour C encircling the nodal singularity

$$\Gamma \equiv \oint_C \boldsymbol{v}\cdot\mathrm{d}\boldsymbol{s}. \tag{2.7}$$

We make use of Stokes' theorem,

$$\Gamma = \iint_S \boldsymbol{\omega} \cdot \mathrm{d}\boldsymbol{S} \tag{2.8}$$

where S is a two-dimensional surface whose boundary is the closed contour C. The wavefunction (2.3) is required to have a single value at each point, so the circulation (2.7) becomes, with the help of (2.4) [29, 31, 39–41],

$$\Gamma = 2\pi n\hbar/m \tag{2.9}$$

where n is an integer. This result informs us that *the circulations are quantized in units of* $2\pi\hbar/m$ *for the state (2.3) in semiclassical cases*. Equation (2.9) is known as *Onsager's quantization of circulations*, in superfluidity [42, 43]. Examples of this quantized vortex will be dealt with in section 4.1 and appendix A.

In hydrodynamics [36–38], the flow satisfying $\boldsymbol{\omega} = 0$ is called potential flow or irrotational flow. Thus the result (2.6) asserts that *the quantum flow is irrotational flow except for the nodal singularities*. The velocity in irrotational flow satisfying (2.6) may be described by the gradient of the velocity potential Φ,

$$\boldsymbol{v} = \nabla\Phi. \tag{2.10}$$

Comparing this with (2.4), we see that

$$\Phi = S/m \tag{2.11}$$

in semiclassical cases. The right-hand side here is undefined to the extent of an arbitrary additive real constant. On substituting (2.10) in (2.7), the circulation becomes

$$\Gamma = [\Phi]_C \tag{2.12}$$

$[\Phi]_C$ being the change in Φ round the closed contour C. Applied to the velocity potential (2.11) it gives (2.9) again.

We now proceed to study only the two-dimensional or plane flow. Let us consider the velocity (2.2) which is solenoidal, namely

$$\nabla \cdot \boldsymbol{v} \equiv \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0. \tag{2.13}$$

The velocity in two-dimensional flow satisfying (2.13) may be described by the rotation of the stream function Ψ,

$$v_x = \frac{\partial \Psi}{\partial y} \qquad v_y = -\frac{\partial \Psi}{\partial x}. \tag{2.14}$$

The magnitude of the vorticity $\boldsymbol{\omega}$ satisfies, with the help of (2.14),

$$\omega = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} = -\nabla^2\Psi \tag{2.15}$$

where ∇^2 is written for the two-dimensional Laplacian operator $\partial^2/\partial x^2 + \partial^2/\partial y^2$. The solenoidal condition (2.13) holds for incompressible fluids in hydrodynamics, since ρ is a constant. In quantum mechanics, however, the probability density (1.1) depends generally on $\boldsymbol{r}$, so the velocity (2.2) is not always solenoidal.

Further, in the two-dimensional irrotational flow, by combining (2.14) with (2.10) we obtain

$$v_x = \frac{\partial \Phi}{\partial x} = \frac{\partial \Psi}{\partial y} \qquad v_y = \frac{\partial \Phi}{\partial y} = -\frac{\partial \Psi}{\partial x}.$$

These are known as the Cauchy–Riemann equations between the velocity potential and the stream function. We can therefore take the complex velocity potential

$$W(z) = \Phi(x, y) + \mathrm{i}\Psi(x, y) \tag{2.16}$$

which is a regular function of the complex variable $z = x + \mathrm{i}y$. The differentiation of $W(z)$ gives us the complex velocity

$$\frac{\mathrm{d}W}{\mathrm{d}z} = v_x - \mathrm{i}v_y. \tag{2.17}$$

In this way we know that in the two-dimensional irrotational flow it is advantageous to use the theory of functions of a complex variable [36–38]. For example, the flow round the angle π/a is expressed by the complex velocity potential [36–38]

$$W = Az^a \tag{2.18}$$

A being a number. With $a = 1$, this expresses the uniform flow of the two-dimensional plane wave in quantum mechanics (section A.1 of appendix A). There are some elementary examples of complex velocity potentials which express the two-dimensional flows in quantum mechanics (see appendix A).

The above-mentioned method will be applied to the flows of the 2D PPB in section 4.

3. The 2D parabolic potential barrier

The Hamiltonian of the 2D isotropic PPB is

$$\hat{H} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + V_0 - \frac{1}{2}m\gamma^2(x^2 + y^2) \tag{3.1}$$

where $V_0 \in \mathbb{R}$ is the maximum potential energy, $m > 0$ is the mass and $\gamma > 0$ is proportional to the square root of the curvature at $(x, y) = (0, 0)$.

A state is represented by a wavefunction $U(x, y)$ satisfying the Schrödinger equation, which now reads, with $\hat{H}$ given by (3.1),

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)U(x, y) + \left\{V_0 - \frac{1}{2}m\gamma^2(x^2 + y^2)\right\}U(x, y) = EU(x, y). \tag{3.2}$$

The energy eigenvalues of (3.2) will be the sum of the energy eigenvalues of the 1D PPB in the x- and y-directions, respectively, i.e.

$$E_{n_x n_y} = E_{n_x} + E_{n_y} \tag{3.3}$$

and the eigenfunctions belonging to these energy eigenvalues will be the product of their corresponding eigenfunctions

$$U_{n_x n_y}(x, y) = u_{n_x}(x)u_{n_y}(y). \tag{3.4}$$

With the notation of preceding papers [11, 12], the energy eigenvalues of the 1D PPB are

$$E^{\pm}_{n_x} = \tfrac{1}{2}V_0 \mp \mathrm{i}(n_x + \tfrac{1}{2})\hbar\gamma \qquad (n_x = 0, 1, 2, \ldots) \tag{3.5}$$

and the corresponding eigenfunctions are

$$u^{\pm}_{n_x}(x) = \mathrm{e}^{\pm\mathrm{i}\beta^2x^2/2}H^{\pm}_{n_x}(\beta x) \qquad \left(\beta \equiv \sqrt{m\gamma/\hbar}\right) \tag{3.6}$$

where $H^{\pm}_{n_x}(\beta x)$ are the polynomials of degree n_x, and the numerical coefficients are discarded. The eigenfunctions $u^{\pm}_{n_x}$ are generalized functions in $\mathcal{S}(\mathbb{R})^{\times}$ of the following Gel'fand triplet:

$$\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})^{\times} \tag{3.7}$$

where $L^2(\mathbb{R})$ is a Lebesgue space and $\mathcal{S}(\mathbb{R})$ is a Schwartz space. The paper [11] also shows that the index + means only outward moving particles and the index − means only inward moving particles.

Thus the results (3.3) and (3.4) of the 2D PPB separate into four types:

Type 1. $E^{++}_{n_x n_y} = E^+_{n_x} + E^+_{n_y} = V_0 - \mathrm{i}(n_x + n_y + 1)\hbar\gamma$

$U^{++}_{n_x n_y}(x, y) = u^+_{n_x}(x)u^+_{n_y}(y) = \mathrm{e}^{+\mathrm{i}\beta^2(x^2+y^2)/2} H^+_{n_x}(\beta x) H^+_{n_y}(\beta y).$

Type 2. $E^{+-}_{n_x n_y} = E^+_{n_x} + E^-_{n_y} = V_0 - \mathrm{i}(n_x - n_y)\hbar\gamma$

$U^{+-}_{n_x n_y}(x, y) = u^+_{n_x}(x)u^-_{n_y}(y) = \mathrm{e}^{+\mathrm{i}\beta^2(x^2-y^2)/2} H^+_{n_x}(\beta x) H^-_{n_y}(\beta y).$

Type 3. $E^{-+}_{n_x n_y} = E^-_{n_x} + E^+_{n_y} = V_0 + \mathrm{i}(n_x - n_y)\hbar\gamma$

$U^{-+}_{n_x n_y}(x, y) = u^-_{n_x}(x)u^+_{n_y}(y) = \mathrm{e}^{-\mathrm{i}\beta^2(x^2-y^2)/2} H^-_{n_x}(\beta x) H^+_{n_y}(\beta y).$

Type 4. $E^{--}_{n_x n_y} = E^-_{n_x} + E^-_{n_y} = V_0 + \mathrm{i}(n_x + n_y + 1)\hbar\gamma$

$U^{--}_{n_x n_y}(x, y) = u^-_{n_x}(x)u^-_{n_y}(y) = \mathrm{e}^{-\mathrm{i}\beta^2(x^2+y^2)/2} H^-_{n_x}(\beta x) H^-_{n_y}(\beta y).$

These eigenfunctions $U^{++}_{n_x n_y}$, $U^{+-}_{n_x n_y}$, $U^{-+}_{n_x n_y}$ and $U^{--}_{n_x n_y}$ are also generalized functions in $\mathcal{S}(\mathbb{R}^2)^{\times}$ of the Gel'fand triplet

$$\mathcal{S}(\mathbb{R}^2) \subset L^2(\mathbb{R}^2) \subset \mathcal{S}(\mathbb{R}^2)^{\times} \tag{3.8}$$

instead of (3.7). Note that the eigenfunctions of types 4 and 3 are conjugate complex functions of types 1 and 2, respectively, i.e.

$$U^{\pm\pm}_{n_x n_y}(x, y)^* = U^{\mp\mp}_{n_x n_y}(x, y)$$

and

$$U^{\pm\mp}_{n_x n_y}(x, y)^* = U^{\mp\pm}_{n_x n_y}(x, y).$$

3.1. Diverging and converging flows

Let us consider first types 1 and 4. In this case the energy eigenvalues $E^{\pm\pm}_{n_x n_y}$ are always complex numbers and the time factors corresponding to them are

$$\mathrm{e}^{-\mathrm{i}E^{\pm\pm}_{n_x n_y} t/\hbar} = \mathrm{e}^{-\mathrm{i}V_0 t/\hbar} \mathrm{e}^{\mp(n_x+n_y+1)\gamma t}.$$

Thus the solutions of type 1 are well defined when $t > 0$, and those of type 4 are well defined when $t < 0$, according to the time boundary condition that time factors of an unstable system are square integrable [11]. Also, $U^{++}_{n_x n_y}(x, y)$ represent particles moving outward from the centre as in figure 1, and $U^{--}_{n_x n_y}(x, y)$ represent particles moving inward to the centre as in figure 2. Thus we shall call these types *diverging* and *converging flows*, respectively. Note that a time reversal occurs, resulting in the interchange of the diverging and converging flows.

For $n_x = n_y = 0$, we obtain the energy eigenvalue

$$E^{\pm\pm}_{00} = V_0 \mp \mathrm{i}\hbar\gamma \tag{3.9}$$

and only one eigenfunction

$$U^{\pm\pm}_{00}(x, y) = \mathrm{e}^{\pm\mathrm{i}\beta^2(x^2+y^2)/2} \tag{3.10}$$

respectively. For $n_x + n_y = 1$, namely $n_x = 1, n_y = 0$ and $n_x = 0, n_y = 1$, we obtain

$$E^{\pm\pm}_{10} = E^{\pm\pm}_{01} = V_0 \mp 2\mathrm{i}\hbar\gamma \tag{3.11}$$

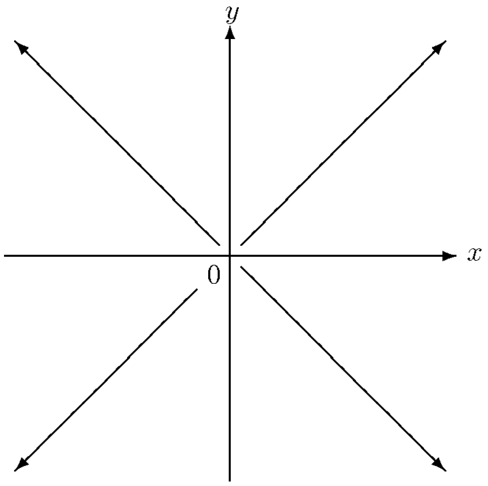

Figure 1. Diverging flows.

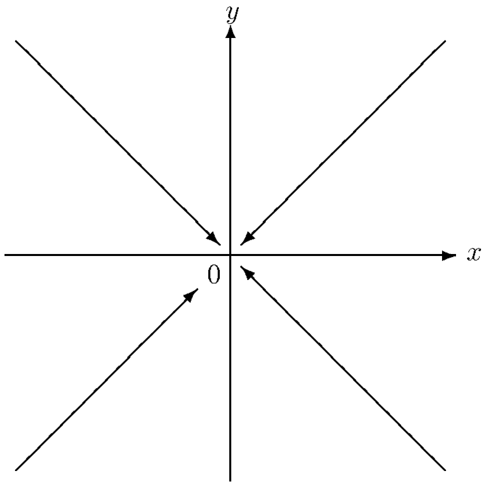

Figure 2. Converging flows.

and two eigenfunctions

$$\begin{aligned} U_{10}^{\pm\pm}(x, y) &= 2\beta x \mathrm{e}^{\pm \mathrm{i}\beta^2(x^2+y^2)/2} \\ U_{01}^{\pm\pm}(x, y) &= 2\beta y \mathrm{e}^{\pm \mathrm{i}\beta^2(x^2+y^2)/2}. \end{aligned} \tag{3.12}$$

There is a twofold degenerate state of types 1 and 4 with $n_x + n_y = 1$. For $n_x + n_y = 2$, namely $n_x = 2, n_y = 0$; $n_x = 1, n_y = 1$; $n_x = 0, n_y = 2$, we obtain

$$E_{20}^{\pm\pm} = E_{11}^{\pm\pm} = E_{02}^{\pm\pm} = V_0 \mp 3\mathrm{i}\hbar\gamma \tag{3.13}$$

and three eigenfunctions

$$\begin{aligned} U_{20}^{\pm\pm}(x, y) &= (4\beta^2 x^2 \mp 2\mathrm{i})\mathrm{e}^{\pm \mathrm{i}\beta^2(x^2+y^2)/2} \\ U_{11}^{\pm\pm}(x, y) &= 4\beta^2 xy \mathrm{e}^{\pm \mathrm{i}\beta^2(x^2+y^2)/2} \\ U_{02}^{\pm\pm}(x, y) &= (4\beta^2 y^2 \mp 2\mathrm{i})\mathrm{e}^{\pm \mathrm{i}\beta^2(x^2+y^2)/2}. \end{aligned} \tag{3.14}$$

There is a threefold degenerate state of types 1 and 4 with $n_x + n_y = 2$. Generally, there is an $(n + 1)$-fold degenerate state of types 1 and 4 with $n_x + n_y = n$. This result is just the same degree of degeneracy as the 2D HO.

For the further discussion of the state of types 1 and 4, we now pass from the Cartesian coordinates x, y to the two-dimensional polar coordinates ρ, φ by means of the equations

$$\begin{aligned} x &= \rho \cos\varphi \\ y &= \rho \sin\varphi. \end{aligned} \tag{3.15}$$

If in the new coordinates we superpose the above-mentioned eigenstates with suitable weights, the result will be the eigenstates of orbital angular momentum $\hat{L}$ defined by

$$\hat{L} = -\mathrm{i}\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = -\mathrm{i}\hbar\frac{\partial}{\partial\varphi}. \tag{3.16}$$

For $n \equiv n_x + n_y = 0$, the eigenfunction (3.10) will be

$$U_{0_0}^{\pm\pm}(\rho, \varphi) \equiv U_{00}^{\pm\pm}(\rho, \varphi) = \mathrm{e}^{\pm \mathrm{i}\beta^2\rho^2/2} \tag{3.17}$$

the suffix m_n of $U_{m_n}^{\pm\pm}$ being the eigenvalue of $\hat{L}/\hbar$. Thus $U_{0_0}^{\pm\pm}(\rho, \varphi)$ is independent of φ and has zero orbital angular momentum. For $n = 1$, a linear combination of the eigenfunctions (3.12) gives

$$\begin{aligned} U_{1_1}^{\pm\pm}(\rho, \varphi) &\equiv U_{10}^{\pm\pm}(\rho, \varphi) + \mathrm{i}U_{01}^{\pm\pm}(\rho, \varphi) = 2\beta\rho \mathrm{e}^{\pm \mathrm{i}\beta^2\rho^2/2}\mathrm{e}^{\mathrm{i}\varphi} \\ U_{-1_1}^{\pm\pm}(\rho, \varphi) &\equiv U_{10}^{\pm\pm}(\rho, \varphi) - \mathrm{i}U_{01}^{\pm\pm}(\rho, \varphi) = 2\beta\rho \mathrm{e}^{\pm \mathrm{i}\beta^2\rho^2/2}\mathrm{e}^{-\mathrm{i}\varphi}. \end{aligned} \tag{3.18}$$

These states are eigenstates of $\hat{L}$ with eigenvalues $\hbar$ and $-\hbar$, respectively. For $n = 2$, (3.14) give

$$\begin{aligned} U_{2_2}^{\pm\pm}(\rho, \varphi) &\equiv U_{20}^{\pm\pm}(\rho, \varphi) + 2\mathrm{i}U_{11}^{\pm\pm}(\rho, \varphi) - U_{02}^{\pm\pm}(\rho, \varphi) = 4\beta^2\rho^2 \mathrm{e}^{\pm \mathrm{i}\beta^2\rho^2/2}\mathrm{e}^{2\mathrm{i}\varphi} \\ U_{0_2}^{\pm\pm}(\rho, \varphi) &\equiv U_{20}^{\pm\pm}(\rho, \varphi) + U_{02}^{\pm\pm}(\rho, \varphi) = 4(\beta^2\rho^2 \mp \mathrm{i})\mathrm{e}^{\pm \mathrm{i}\beta^2\rho^2/2} \\ U_{-2_2}^{\pm\pm}(\rho, \varphi) &\equiv U_{20}^{\pm\pm}(\rho, \varphi) - 2\mathrm{i}U_{11}^{\pm\pm}(\rho, \varphi) - U_{02}^{\pm\pm}(\rho, \varphi) = 4\beta^2\rho^2 \mathrm{e}^{\pm \mathrm{i}\beta^2\rho^2/2}\mathrm{e}^{-2\mathrm{i}\varphi}. \end{aligned} \tag{3.19}$$

These states are also eigenstates of $\hat{L}$ with eigenvalues $2\hbar$, 0 and $-2\hbar$. Substituting these eigenfunctions in (1.2), we obtain the two-dimensional polar coordinates $j_{m_n\rho}^{\pm\pm}$, $j_{m_n\varphi}^{\pm\pm}$ of $\boldsymbol{j}_{m_n}^{\pm\pm}$, which are the probability currents of the states $U_{m_n}^{\pm\pm}$. The result is

$$j_{0_0\rho}^{\pm\pm}(t, \rho, \varphi) = \pm \mathrm{e}^{\mp 2\gamma t}\gamma\rho \qquad j_{0_0\varphi}^{\pm\pm}(t, \rho, \varphi) = 0 \tag{3.20}$$

$$\begin{aligned} j_{1_1\rho}^{\pm\pm}(t, \rho, \varphi) &= \pm 4\mathrm{e}^{\mp 4\gamma t}\gamma\beta^2\rho^3 \qquad & j_{1_1\varphi}^{\pm\pm}(t, \rho, \varphi) &= 4\mathrm{e}^{\mp 4\gamma t}\gamma\rho \\ j_{-1_1\rho}^{\pm\pm}(t, \rho, \varphi) &= \pm 4\mathrm{e}^{\mp 4\gamma t}\gamma\beta^2\rho^3 \qquad & j_{-1_1\varphi}^{\pm\pm}(t, \rho, \varphi) &= -4\mathrm{e}^{\mp 4\gamma t}\gamma\rho \end{aligned} \tag{3.21}$$

$$\begin{aligned} j_{2_2\rho}^{\pm\pm}(t, \rho, \varphi) &= \pm 16\mathrm{e}^{\mp 6\gamma t}\gamma\beta^4\rho^5 \qquad & j_{2_2\varphi}^{\pm\pm}(t, \rho, \varphi) &= 32\mathrm{e}^{\mp 6\gamma t}\gamma\beta^2\rho^3 \\ j_{0_2\rho}^{\pm\pm}(t, \rho, \varphi) &= \pm 16\mathrm{e}^{\mp 6\gamma t}\gamma\rho(\beta^4\rho^4 + 3) \qquad & j_{0_2\varphi}^{\pm\pm}(t, \rho, \varphi) &= 0 \\ j_{-2_2\rho}^{\pm\pm}(t, \rho, \varphi) &= \pm 16\mathrm{e}^{\mp 6\gamma t}\gamma\beta^4\rho^5 \qquad & j_{-2_2\varphi}^{\pm\pm}(t, \rho, \varphi) &= -32\mathrm{e}^{\mp 6\gamma t}\gamma\beta^2\rho^3. \end{aligned} \tag{3.22}$$

Thus $\boldsymbol{j}_{m_n}^{\pm\pm}$ depend on the time t. There is a similar procedure for large values of n.

3.2. Corner flows

Let us now study types 2 and 3. In this case the energy eigenvalues $E_{n_x n_y}^{\pm\mp}$ are also in general complex numbers, but with the striking difference that all the eigenstates belonging to each energy eigenvalue are infinitely degenerate. The corresponding time factors are

$$\mathrm{e}^{-\mathrm{i}E_{n_x n_y}^{\pm\mp}t/\hbar} = \mathrm{e}^{-\mathrm{i}V_0 t/\hbar}\mathrm{e}^{\mp(n_x - n_y)\gamma t}.$$

Thus the solutions of type 2 are well defined when $t > 0$, and those of type 3 are well defined when $t < 0$, for the case of $n_x > n_y$, and vice versa. Also, $U_{n_x n_y}^{+-}(x, y)$ represent particles which, coming from the y-direction, round the centre and go off to the x-direction as in figure 3, and $U_{n_x n_y}^{-+}(x, y)$ represent particles which, coming from the x-direction, round the centre and go off to the y-direction as in figure 4. Note that a time reversal occurs, resulting in the interchange of these corner flows.

Stationary flows. The above time factors now show that for the case of $n_x = n_y$, there are *stationary flows*. For $n_x = n_y \equiv n = 0, 1, 2, \ldots$, the energy eigenvalues associated with stationary flows are the same real number:

$$E_{nn}^{\pm\mp} = V_0. \tag{3.23}$$

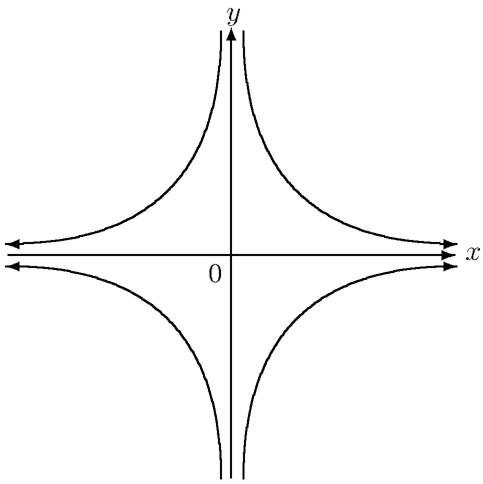

Figure 3. Corner flows moving from the y- to the x-direction.

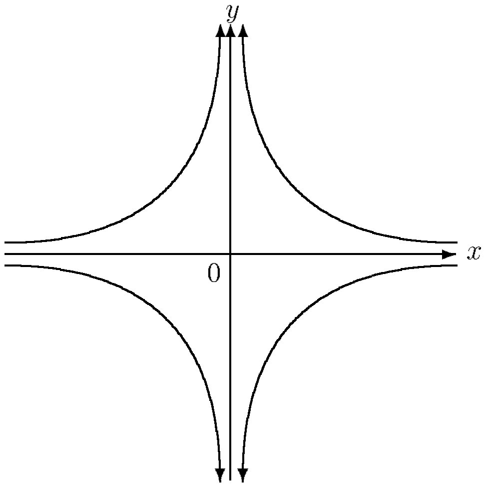

Figure 4. Corner flows moving from the x- to the y-direction.

The first few infinitely degenerate eigenfunctions belonging to this energy eigenvalue (3.23) are

$$\begin{aligned}
&U_{00}^{\pm\mp}(x, y) = \mathrm{e}^{\pm \mathrm{i}\beta^2(x^2-y^2)/2} \\
&U_{11}^{\pm\mp}(x, y) = 4\beta^2 xy\mathrm{e}^{\pm \mathrm{i}\beta^2(x^2-y^2)/2} \\
&U_{22}^{\pm\mp}(x, y) = 4[4\beta^4x^2y^2 + 1 \pm 2\mathrm{i}\beta^2(x^2 - y^2)]\mathrm{e}^{\pm \mathrm{i}\beta^2(x^2-y^2)/2} \\
&\ldots .
\end{aligned} \tag{3.24}$$

For the further study of the stationary flows of the 2D PPB with the Hamiltonian (3.1), it is convenient to make a transformation to the rectangular hyperbolic coordinates u, v, given by

$$\begin{aligned}
&u = x^2 - y^2 \\
&v = 2xy.
\end{aligned} \tag{3.25}$$

The eigenfunctions (3.24) will become in the new representation

$$\begin{aligned}
&U_{00}^{\pm\mp}(u, v) = \mathrm{e}^{\pm \mathrm{i}\beta^2 u/2} \\
&U_{11}^{\pm\mp}(u, v) = 2\beta^2 v\mathrm{e}^{\pm \mathrm{i}\beta^2 u/2} \\
&U_{22}^{\pm\mp}(u, v) = 4(\beta^4v^2 + 1 \pm 2\mathrm{i}\beta^2 u)\mathrm{e}^{\pm \mathrm{i}\beta^2 u/2} \\
&\ldots .
\end{aligned} \tag{3.26}$$

The factors $\mathrm{e}^{\pm \mathrm{i}\beta^2 u/2}$ occurring in (3.26) describe plane waves in the uv-plane, i.e. the motion of the wave $\mathrm{e}^{\mathrm{i}\beta^2 u/2}$ is in the direction specified by figure 3 and that of the wave $\mathrm{e}^{-\mathrm{i}\beta^2 u/2}$ is in the direction specified by figure 4. These eigenfunctions substituted in (1.2) give the rectangular hyperbolic coordinates $j_{nnu}^{\pm\mp}$, $j_{nnv}^{\pm\mp}$ of $\boldsymbol{j}_{nn}^{\pm\mp}$, which are the probability currents of the states $U_{nn}^{\pm\mp}$. They give

$$\begin{aligned}
&j_{00u}^{\pm\mp}(u, v) = \pm\gamma h_u/2 \qquad j_{00v}^{\pm\mp}(u, v) = 0 \\
&j_{11u}^{\pm\mp}(u, v) = \pm 2\gamma\beta^4 v^2 h_u \qquad j_{11v}^{\pm\mp}(u, v) = 0 \\
&j_{22u}^{\pm\mp}(u, v) = \pm 8\gamma[(\beta^4v^2 + 5)(\beta^4v^2 + 1) + 4\beta^4u^2]h_u \\
&j_{22v}^{\pm\mp}(u, v) = \mp 64\gamma\beta^4 uvh_v \\
&\ldots
\end{aligned} \tag{3.27}$$

where the scale factors $h_u = h_v = 2\sqrt[4]{u^2 + v^2}$. Thus $j_{nn}^{\pm\mp}$ can never depend on the time t. We see from this result the suitability of the term 'stationary flows'.

4. Hydrodynamical formulation of the 2D parabolic potential barrier

In the present section we shall apply the hydrodynamical formulation of quantum mechanics reviewed in section 2 to the 2D PPB flows dealt with in section 3, in section 4.1 the velocities and the vortices of diverging and converging flows studied in section 3.1 and in section 4.2 the velocities, the vortices and the complex velocity potentials of stationary flows as special cases of corner flows studied in section 3.2.

4.1. Vortices of diverging and converging flows

The non-vanishing φ-components of the probability currents (3.21), the first and third of equations (3.22), show that for the eigenstates of orbital angular momentum there are vortices around the origin $\rho = 0$. Note that this point $\rho = 0$ is the node of the corresponding wavefunctions (3.18), (3.19). To obtain an understanding of the physical features of these vortices it is better to work with the velocity defined by (2.2) and the circulation defined by (2.7). For the states (3.17)–(3.19), we obtain the following velocities:

$$v_{0_0\rho}^{\pm\pm} = \pm\gamma\rho \qquad v_{0_0\varphi}^{\pm\pm} = 0 \tag{4.1}$$

$$\begin{aligned} v_{1_1\rho}^{\pm\pm} &= \pm\gamma\rho & v_{1_1\varphi}^{\pm\pm} &= \tfrac{\hbar}{m\rho} \\ v_{-1_1\rho}^{\pm\pm} &= \pm\gamma\rho & v_{-1_1\varphi}^{\pm\pm} &= -\tfrac{\hbar}{m\rho} \end{aligned} \tag{4.2}$$

$$\begin{aligned} v_{2_2\rho}^{\pm\pm} &= \pm\gamma\rho & v_{2_2\varphi}^{\pm\pm} &= \tfrac{2\hbar}{m\rho} \\ v_{0_2\rho}^{\pm\pm} &= \pm\gamma\rho\tfrac{\beta^4\rho^4+3}{\beta^4\rho^4+1} & v_{0_2\varphi}^{\pm\pm} &= 0 \\ v_{-2_2\rho}^{\pm\pm} &= \pm\gamma\rho & v_{-2_2\varphi}^{\pm\pm} &= -\tfrac{2\hbar}{m\rho}. \end{aligned} \tag{4.3}$$

Note that the nodes of (3.17)–(3.19) go over into the singularities of the corresponding φ-components of the above velocities. Now the circulation (2.7), round a closed contour C encircling the singularity at the origin $\rho = 0$, reads

$$\Gamma_{m_n}^{\pm\pm} = \oint_C v_{m_n\varphi}^{\pm\pm}\,\mathrm{d}s. \tag{4.4}$$

On substituting (4.1)–(4.3) in (4.4) we obtain the following formula:

$$\Gamma_{m_n}^{\pm\pm} = 2\pi\frac{m_n\hbar}{m} \tag{4.5}$$

where m_n on the right-hand side of (4.5) is one of the values $n, n-2, n-4, \ldots, -n$. These are in agreement with expression (2.9) of section 2, and they show that, *for the eigenstates (3.17)–(3.19) of orbital angular momentum, the circulations of the 2D PPB are quantized.* One could make calculations for large values of n and one would be led to the same results.

Comparing (4.5) with (A.14) of appendix A, we see that the vorticities of these states must be of the form (A.11). This leads to the result that the velocity potentials of the form (A.12) can exist except the singularity $\rho = 0$. However, the stream functions of these states do not exist, since the velocities (4.1)–(4.3) cannot be solenoidal. As an example we try to calculate the divergence (2.13) for (4.1). The result is

$$\nabla\cdot\boldsymbol{v}_{0_0}^{\pm\pm} \equiv \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho v_{0_0\rho}^{\pm\pm}) + \frac{1}{\rho}\frac{\partial}{\partial\varphi}v_{0_0\varphi}^{\pm\pm} = \pm 2\gamma \neq 0.$$

Now the $\pm$ sign shows that $U_{0_0}^{++}$ is connected with the diverging flow and $U_{0_0}^{--}$ is connected with the converging one. This result follows from the fact that the probability densities of these flows depend on t in (2.1). Therefore we cannot obtain the complex velocity potentials defined by (2.16). This result also holds for large values of n.

4.2. Vortices and complex velocity potentials of stationary flows

Taking into account that a node of the wavefunction $\psi(t, \boldsymbol{r})$ derives a singularity of the velocity (2.2) of section 2 which is divided by $|\psi(t, \boldsymbol{r})|^2$, one may consider that vortices exist at the nodal singularities of the wavefunction [29, 31]. As noted in section 3.2, the solutions involving the stationary states are infinitely degenerate. One can therefore make up the wavefunction which has a countable number of nodes at any points in terms of the superposition of the infinitely degenerate states.

Complex velocity potentials of stationary flows. The probability currents (3.27) now show that, for the case of $n = 0$ and 1, there are stationary flows which move along the hyperbolae (each line with v constant). To obtain an understanding of the physical features of these flows it is better to work with the velocity defined by (2.2) and the complex velocity potential defined by (2.16). For $n = 0$ and 1, the velocities give the same result

$$v_u^{\pm\mp} = \pm\tfrac{1}{2}\gamma h_u \qquad v_v^{\pm\mp} = 0. \tag{4.6}$$

Note that the nodal singularity $v = 0$ of the second of equations (3.26) does not appear in these velocities. Thus we can take their rotation, so that we obtain the vorticity (2.5)

$$\omega^{\pm\mp} \equiv h_u h_v \left[\frac{\partial}{\partial u}\left(\frac{v_v^{\pm\mp}}{h_v}\right) - \frac{\partial}{\partial v}\left(\frac{v_u^{\pm\mp}}{h_u}\right)\right] = 0.$$

These equations are the general result (2.6), and therefore the velocity potentials defined by (2.10) must exist. If we transform to rectangular hyperbolic coordinates u, v, equations (2.10) become

$$v_u = h_u \frac{\partial \Phi}{\partial u} \qquad v_v = h_v \frac{\partial \Phi}{\partial v} \tag{4.7}$$

and the velocity potentials for $n = 0$ and 1 are thus

$$\Phi^{\pm\mp} = \pm\tfrac{1}{2}\gamma u. \tag{4.8}$$

Note that they are proportional to the phase factors of (3.26). Further, the divergence (2.13) gives

$$\nabla \cdot \boldsymbol{v}^{\pm\mp} \equiv h_u h_v \left[\frac{\partial}{\partial u}\left(\frac{v_u^{\pm\mp}}{h_v}\right) + \frac{\partial}{\partial v}\left(\frac{v_v^{\pm\mp}}{h_u}\right)\right] = 0.$$

Thus the velocities (4.6) are solenoidal, so we can obtain the stream functions defined by (2.14). The equations (2.14) are also expressed, as in equations (4.7),

$$v_u = h_v \frac{\partial \Psi}{\partial v} \qquad v_v = -h_u \frac{\partial \Psi}{\partial u} \tag{4.9}$$

and the stream functions for $n = 0$ and 1 are thus

$$\Psi^{\pm\mp} = \pm\tfrac{1}{2}\gamma v. \tag{4.10}$$

Substituting (4.10) into (2.15) we obtain $\omega^{\pm\mp} = 0$ again. For the states represented by the first and second of equations (3.26), the complex velocity potential (2.16) gives, from (4.8) and (4.10),

$$W^{\pm\mp} = \pm\frac{1}{2}\gamma u \pm \frac{\mathrm{i}}{2}\gamma v$$
$$= \pm\tfrac{1}{2}\gamma z^2 \tag{4.11}$$

since $z^2 = u + \mathrm{i}v$. Equations (4.11) are of the form (2.18) with $a = 2$, and they show that, *for the stationary states (3.26) with $n = 0$ and 1, the complex velocity potentials of the 2D PPB express the flows round a right angle*. One could make calculations in terms of Cartesian coordinates and one would be led to the same conclusion.

5. Discussion

We have obtained the exact solutions of the 2D PPB. One class of the solutions is the diverging and converging flows of section 3.1. These solutions always have complex energy eigenvalues and are expressed by generalized eigenfunctions, which mean that the diverging and converging flows are not stationary. These generalized eigenfunctions can be obtained from the eigenfunctions (A.5) of the 2D HO by the analytical continuation, in the same way as the 1D PPB [11]. Again, they can be superposed to give the eigenstates of orbital angular momentum.

The other class of the solutions is the corner flows of section 3.2. All the solutions are infinitely degenerate and involve stationary flows with a real energy eigenvalue. It should be noted that there are no stationary flows in the 1D or 3D *isotropic* PPB. For $n = 0$ and 1 in the stationary flows, we have found the flows round a right angle that are expressed by the complex velocity potentials (4.11) of section 4.2, but for $n \geqslant 2$ the complex velocity potentials do not exist, because the imaginary parts of the polynomials $H^{\pm}_{n_x}(\beta x)$ in (3.6) cause the streamlines to depart from hyperbolae. One may, however, find a new flow as the result of a kind of superposition of the infinitely degenerate states.

One would expect to be able to obtain a more direct solution of the eigenvalue problem of the Hamiltonian (3.1) by working all the time in the two-dimensional polar coordinates, instead of working in the Cartesian coordinates and transforming at the end to the two-dimensional polar coordinates, as was done in section 3.1, but under suitable boundary conditions in the two-dimensional polar coordinates one would obtain only the diverging and converging flows of section 3.1, i.e. not the corner flows of section 3.2. It is also pointed out that one can obtain only the diverging and converging flows from the analytical continuation of the solutions of the 2D HO. These facts mean that *the choice of coordinate systems is quite important in the eigenvalue problem of the unstable system in non-relativistic quantum mechanics, since coordinate systems impose a restriction on the symmetry of boundary conditions*. The source of this conclusion lies in the existence of a very large class of solutions for the unstable system.

In the discussion from the hydrodynamical point of view carried out in section 4 the velocity plays an important role, more than the probability current does. Remembering the definition of the velocity (2.2) and also comparing the velocities of (4.1)–(4.3) and (4.6) with the probability currents of (3.20)–(3.22) and (3.27), we see that the velocities have the following two advantages over the probability currents: the first one is the fact that *the velocity has no ambiguity arising from the normalization of the states, while the probability currents cannot avoid it*. The second is that *the velocity does not depend on the time t even for the non-stationary states, whereas the probability current generally depends on t*. It is important that the velocity can be a good observable in these quantum processes, since it has no time dependence. It should also be noticed that the velocities given by (4.1) and (4.6) do not contain any order of $\hbar$ at all. This fact indicates that these flows will have a kind of classical property. Actually from the existence of the stationary flows having a kind of classical property, we have pointed out in another paper [44, 45] that the dynamical system composed of several of these PPBs forms the quasi-stable semiclassical system. In the new scheme for Gel'fand triplet [44] the infinite degeneracy can possibly play an essential role in energy and entropy productions in thermodynamical processes [45]. As noticed at the beginning of section 4.2, those stationary flows with infinite degeneracy can also be the origin of many variety of vortices in two-dimensional space. The study of the vortex structures and motions will open new physical applications in the present model. Furthermore the hydrodynamical approach, in terms of the velocity, vorticity, complex velocity potential and so on, will give us a clearer insight for understanding the quantum flows of PPB. On the other hand the non-stationary solutions will possibly describe growing and decaying vortices, which are also interesting objects to investigate further. We will also be able to describe the time evolution of vortex structures in terms of appropriate superposition of solutions of Gel'fand triplets. It will be an interesting theme to study the hydrodynamical aspect more precisely in the present scheme.

Appendix A. Elementary applications

A.1. The free particle

Let us first consider the free particle in two dimensions as an example of a stationary state. The plane wave is of the form

$$u_{p_x p_y}(x, y) = a\mathrm{e}^{\mathrm{i}(p_x x + p_y y)/\hbar} \tag{A.1}$$

where a is independent of t, x and y. The probability current of the plane wave is

$$j_x(x, y) = |a|^2 p_x/m \qquad j_y(x, y) = |a|^2 p_y/m$$

and hence their velocity is

$$v_x = p_x/m \qquad v_y = p_y/m.$$

Equations (2.6) and (2.13) are easily seen to hold for the plane wave. Therefore the velocity potential satisfying (2.10) or (2.11) of the plane wave is

$$\Phi = (p_x x + p_y y)/m \tag{A.2}$$

and the stream function satisfying (2.14) is

$$\Psi = (p_x y - p_y x)/m. \tag{A.3}$$

For the state represented by (A.1), the complex velocity potential (2.16) gives, from (A.2) and (A.3),

$$\begin{aligned} W &= (p_x x + p_y y)/m + \mathrm{i}(p_x y - p_y x)/m \\ &= (p_x - \mathrm{i}p_y)z/m. \end{aligned} \tag{A.4}$$

Equation (A.4) is of the form (2.18) with $a = 1$, and it shows that *the complex velocity potential of the plane wave just expresses uniform flow*.

A.2. The HO

We shall now consider the eigenstate of the 2D HO as an example of a closed state of the stable system. The eigenfunction is, in terms of the Cartesian coordinates x, y,

$$u_{n_x n_y}(x, y) = N_{n_x} N_{n_y} \mathrm{e}^{-\alpha^2(x^2+y^2)/2} H_{n_x}(\alpha x) H_{n_y}(\alpha y) \qquad \left(\alpha \equiv \sqrt{m\omega/\hbar}\right) \tag{A.5}$$

where N_{n_x}, N_{n_y} are the normalizing constants. Since the Hermite polynomials $H_{n_x}(\alpha x)$, $H_{n_y}(\alpha y)$ are real functions of x, y, respectively the probability current of the HO vanishes and their velocity also vanishes. Therefore the velocity potential and the stream function all vanish. For the state represented by (A.5), the complex velocity potential gives

$$W = 0 \tag{A.6}$$

which expresses *fluid at rest* in hydrodynamics. This fluid at rest, however, is not the only one that is physically permissible for a closed state in quantum mechanics, as we can also have flows which are *vortical*. For these flows the vorticity may contain singularities in the xy-plane. Such flows will be dealt with in section A.3.

A.3. Flows in a central field of force

As a final example we shall consider the bound state in a certain central field of force. The eigenfunction is, in terms of the polar coordinates r, θ, φ,

$$u_{nlm_l}(r, \theta, \varphi) = R_{nl}(r) Y_{lm_l}(\theta, \varphi) \tag{A.7}$$

where the spherical harmonics $Y_{lm_l}(\theta, \varphi)$ are of the form

$$Y_{lm_l}(\theta, \varphi) = C_{lm_l} P_l^{|m_l|}(\cos\theta) \mathrm{e}^{\mathrm{i} m_l \varphi} \tag{A.8}$$

and C_{lm_l} are the normalizing constants. Since $R_{nl}(r)$ for the bound state and the associated Legendre polynomials $P_l^{|m_l|}(\cos\theta)$ are real functions, the polar coordinates j_r, j_θ, j_φ of (1.2) in a central field of force are

$$j_r(r, \theta, \varphi) = j_\theta(r, \theta, \varphi) = 0 \qquad j_\varphi(r, \theta, \varphi) = |u_{nlm_l}(r, \theta, \varphi)|^2 \frac{m_l \hbar}{m r \sin\theta}.$$

In consequence, a simple treatment becomes possible, namely, we may consider the velocity for a definite direction θ and then we can introduce the radius $\rho = r \sin\theta$ in the above equations and obtain a problem in two degrees of freedom ρ, φ. The two-dimensional polar coordinates v_ρ, v_φ of (2.2) give

$$v_\rho = 0 \qquad v_\varphi = \frac{m_l \hbar}{m \rho}.$$

Their divergence readily vanishes. If we transform to two-dimensional polar coordinates ρ, φ, equations (2.14) become

$$v_\rho = \frac{1}{\rho} \frac{\partial \Psi}{\partial \varphi} \qquad v_\varphi = -\frac{\partial \Psi}{\partial \rho} \tag{A.9}$$

and the stream function in a central field of force is thus

$$\Psi = -\frac{m_l \hbar}{m} \log \rho. \tag{A.10}$$

On substituting (A.10) in (2.15) we obtain

$$\omega = \frac{m_l \hbar}{m} \nabla^2 \log \rho = 2\pi \frac{m_l \hbar}{m} \delta(\boldsymbol{\rho}) \tag{A.11}$$

where $\delta(\boldsymbol{\rho})$ is the two-dimensional Dirac δ function. Thus the vorticity in a central field of force vanishes everywhere except the origin $\rho = 0$. This singularity will lie along the quantization axis $\theta = 0$ and π in three-dimensional space. The velocity potential satisfying (2.10) or (2.11) in a central field of force is

$$\Phi = \frac{m_l \hbar}{m} \varphi. \tag{A.12}$$

For the state represented by (A.7), the complex velocity potential (2.16) gives, from (A.12) and (A.10),

$$\begin{aligned} W &= \frac{m_l \hbar}{m}\varphi - \mathrm{i}\frac{m_l \hbar}{m}\log\rho \\ &= -\mathrm{i}\frac{m_l \hbar}{m}\log z \end{aligned} \tag{A.13}$$

since $z = \rho e^{\mathrm{i}\varphi}$. According to hydrodynamics [38], this expresses the *vortex filament*. The strength of the vortex filament is defined by the circulation (2.7) round a closed contour C encircling the singularity at the origin $\rho = 0$. On substituting (A.11) in (2.8) we obtain

$$\Gamma = 2\pi \frac{m_l \hbar}{m} \tag{A.14}$$

where m_l is an integer. This result is to be expected from (A.12) and (2.12). This is of exactly the same form as (2.9), and it shows that *the circulations are quantized for the state (A.7) moving in a central field of force*.

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